Kinematics in One Dimension

Kinematics

- Description of motion without the agents causing or modifying the motion

Motion

- Change of object *position* with *time*, respect to a reference point

Position

- Location of an object with respect to a reference point. Scalar
- $x(t)$

Velocity

- Rate of change of position
- $v(t)$

Acceleration

- Rate of change of velocity
- $a(t)$

Motion in 1D can be described as a **function** of time.

Displacement

- Change in position, through a time interval
- $\Delta x = x_f - x_i$
- Normally a vector, a scalar in 1-D motion
- Positive or negative, depending on what point of reference is used
- Equal to the area **under** the velocity v. time graph
    - $\Delta x(t) = \lim_{\Delta t \to 0} \Sigma v(t_i)\Delta t = \int_{t_i}^{t_f} v(t)dt$

Average Velocity

- Defined as Δx per Δt
- $\vec{v} \equiv \frac{\Delta x}{\Delta t}$
- Or, the slope of the line connecting two points on the position-time graph

Instantaneous Velocity

- The slope of a tangent of a point on the position-time graph
- Really just *average velocity* with two infinitesimally close points
  - $t_2 \to t_1, \Delta t \to 0$ and $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt} \equiv v$
- $v_{ins} = v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$
- Represented by v_{ins}

Acceleration

- Acceleration involves the change in velocity, which includes speeding up and slowing down.
- $a_{x,avg} \equiv \frac{\Delta v_x}{\Delta t} = \frac{v_{x,f} - v_{x,i}}{t_f - t_i}$
- How does acceleration change?
  - Change the magnitude of velocity (increase) otherwise known as "speeding up"
  - Change the magnitude of velocity (decrease) otherwise known as "slowing down"
  - Change the direction of velocity (**does not happen in 1-D motion**)

Average Acceleration

- The slope of the tangent of the velocity-time graph
- Defined as change of velocity per time interval
- $\vec{a} \equiv \frac{\Delta v}{\Delta t}$
- Or, the slope of the line connecting two points on the velocity-time graph

Instantaneous Acceleration

- The slope of a tangent of a point on the velocity-time graph
- Really just *average acceleration* with two infinitesimally close points
  - $t_2 \to t_1, \Delta t \to 0$ and $\frac{\Delta v}{\Delta t} \to \frac{dv}{dt} \equiv a$
- $a_{ins} = a_x = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$
- Inverted: $v_{ins}(t) = \int_{t_1}^{t_2} a_x(t) dt$
- Represented by a_{ins}

Kinematic Equations for CONSTANT ACCELERATION

$v_f = v_i + at$
$x_f = x_i + v_i t + \frac{1}{2} a t^2$
$v_f^2 - v_i^2 = 2a(x_f - x_i)$: found through rearrangement of 1 and 2
$v = v_0 + at$

$x = x_0 + v_0 t + \frac{1}{2} a t^2$
$v^2 - v_0^2 = 2a(x - x_0)$
$x_f = x_i + vt$
$x = x_0 + vt$
$\Delta x = \frac{1}{2} a t^2$

NOTE: VELOCITY AND ACCELERATION CAN BE IN OPPOSITE DIRECTIONS

- All "slowing down" means is that velocity and acceleration have opposing signs

Free Fall

- Objects moving freely under only the influence of gravity

$g \approx 9.81 m/s^2 \quad \text{[down]}$

Conventionally, when working with kinematics equations, replace a with g.

- Free fall is the same for all objects (barring air resistance, eg. feather vs. bowling ball).
- g varies with location and height.
- If air resistance is significant, it is not free fall.

Acceleration on an inclined plane

1. Draw coordinate system on the object, x parallel and y perpendicular to the inclined surface.
2. Draw g downwards from the origin of the coordinate system.
3. Draw the y-component and x-component of the acceleration.
    1. $g_y = gcos(\theta)$ and $g_x = gsin(\theta)$

Kinematics in Two Dimensions

Vectors

Scalar

- A real number with units
- Ordinary or italic font
  Vector
- Described by a scalar (magnitude) and a direction.
- Boldface font or arrow notation

Vector Addition

- eg. $\vec{A} + \vec{B}$
- Triangle Method
  - Connect the start of $\vec{A}$ and $\vec{B}$
- Parallelogram
  - If $\vec{B}$ starts at the same point of $\vec{A}$, forming a paralellogram makes the line connecting the diagonals $\vec{C}$.

Coordinate Systems

- To break down a vector into components:
  - Define axes.
    - Axes don't have to be horizontal and vertical.
  - Draw components of vector.
    - Vector components (eg. v_x, v_y, v_z, are scalars.)

3D Coordinates

- Use a right-handed coordinate system.
- Meaning, the axes go |_ rather than _|.
  - Or use the hand rule.

Unit Vectors

- Vector of magnitude 1, no units, direction of the original vector. Simply a way to signify which axis a vector is on.
- Denoted as $\hat{a}$. Coordinate vectors for x, y, z dimensions are $\hat{\imath}, \hat{\jmath}, \hat{k}$
  - eg. $\vec{A} + \vec{B} = \vec{C}$
  - $(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) = (C_x\hat{\imath} + C_y\hat{\jmath} + C_z\hat{k})$
    - So, $A_x\hat{\imath} + B_x\hat{\imath} = C_x\hat{\imath}$
  - Or:
    - $(A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} = (C_x + C_y)$
- To subtract vectors, flip the direction of the negative vector.

Position

- The trajectory is the path of the particle.
- The position vector ($\vec{r}$) points from the origin to the particle's location at that time.
  - The coordinates of the plane (x, y) are the components of the position vector.

- Position and velocity can be broken into unit vector components
- $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$
- Notational:
  - $\vec{v} = \dot{\vec{r}}$
  - $\vec{v} = \dot{\vec{r}}$
- All constant acceleration equations need to split into x and y components

Projectile Motion

- Assumes:
  - The free-fall acceleration is constant
    - It is downwards
      - (As long as the distance is relatively small to where the Earth's curvature is negligible)
  - Air resistance is negligible
- A straight line vertical velocity would be $\vec{v}t$, so $\frac{1}{2}\vec{g}t$ is the amount the projectile falls from that straight line velocity as a function of t.

Uniform Circular Motion

- Object moving in a circle with constant speed but not constant velocity
  - $\vec{v}$ is always tangential to the path
- $\Delta\vec{v}$ is perpendicular to $\Delta\vec{r}$
- $a_c = \frac{v^2}{r}$ with a_c in the direction of the center of the circle
  - The tangential (causes change in magnitude of speed) component of acceleration is $a_t = \frac{d|\vec{v}|}{dt}$. Direction is same as $\vec{v}$ if the speed is increasing, opposite if speed is decreasing
  - The radial (centripetal) component arises from change in direction, the perpendicular one, $a_r = -a_c$
- For non-uniform circular motion:
  - $\Delta\vec{v}$ has parallel and perpendicular components to $\Delta\vec{r}$
- Period, T, is time for one revolution
  - Speed = $\frac{2\pi r}{T}$ so $T = \frac{2\pi r}{v}$
  - Also $T = \frac{2\pi}{\omega}$ where ω is angular velocity

Relative Motion

- Two axes with separate axes A, B.
- $v\vec{_{PA}} = v\vec{_{PB}} + v\vec{_{BA}}$

- $\vec{v_{PA}}$ is the velocity of an object at P from the perspective of A
- $\vec{v_{PB}}$ is the velocity of an object at P from the perspective of B

- If there are two reference frames with axes x, y and x', y'
  - The relative displacement of the two axes is $\mathbf{r_0}$
  - $\vec{r} = \vec{r}' + \vec{r_0}$
  - $\vec{v} = \vec{v}' + \vec{v_0}$
  - $\vec{a} = \vec{a}'$

Forces

- Forces are the cause of acceleration, not movement
- Units of $kg \cdot \frac{m}{s^2}$ or N. Is a vector
- A force's agent is the object that applies the force
- Either contact or contact-free (long-range force or field force)
  - Contact
    - eg. Normal force, friction, air resistance, bouyancy
  - Non-contact
    - eg. Gravity, electromagnetic, weak (eg. radiation) & strong forces (eg. forces that hold atoms together)
- Newton's First Law
  - An isolated object, free form external force, will continue at constant velocity.
- Newton's Second Law
  - $\vec{F}_{net} = m\vec{a}$
  - $\vec{F}_{net} = F_{netx}\hat{\imath} + F_{nety}\hat{\jmath} + F_{netz}\hat{z}$
- The magnitude of the gravitational force on YOU is **weight**. Units: newtons
  - Weight $\propto$ mass
- $F = G\frac{Mm}{r^2}$
  - Force of gravity between masses
  - r is the distance between the masses
  - M and m are the masses of the two masses
- Newton's Third Law
  - For every force there is an equal but opposite force
  - The pair of forces **ALWAYS** act on different objects, or else the forces would cancel each other out.
- FBD
  - Pick one object (the 'body')
  - Identify all external forces that act DIRECTLY on the body

        - Set a coordinate system
        - Represent the object as a dot at the origin
        - Draw arrows
        - Draw components of each force
        - Indicate direction of acceleration
    - Forces:
        - Weight, $\vec{w}$
        - Spring force, $\vec{F}_s$
        - Tension, $\vec{T}$
        - Normal force, $\vec{N}$
        - Friction
            - $\vec{f}_s$
            - $\vec{f}_k$
        - Drag, $\vec{D}$
        - Thrust, $\vec{F}_{thrust}$
        - Electric & magnetic forces, $\vec{F}_E$, $\vec{F}_M$
    - Friction
        - The force that resists sliding of surfaces against each other
        - Static friction $\vec{F}_s$ resists movement while stationary
        - Kinetic friction $\vec{F}_k$ resists movement while moving
        - $f_s \leq \mu_s n$, where μ_s is the friction coefficient and n is the magnitude of the normal force. $\mu_s n$ also can be called $f_{s,max}$
        - Usually, $\mu_k < \mu_s$
    - Equilibrium
        - Special case where $\vec{a} = 0$
        - **The vector sum of all forces acting on the object is 0**
        - $F_{net_x} = 0, F_{net_y} = 0, F_{net_z} = 0$
    - How to do $F = ma$ questions
        - Draw free body diagram CAREFULLY
        - Find direction of acceleration if necessary
    - Circular motion
        - Motion follows a circular path, with constant speed. Velocity is not constant, and therefore acceleration is not constant
        - Velocity is always tangential to the circle and acceleration is always pointing to the center of the circle
            - Angular velocity, $\omega = v/r$

        - Centripetal acceleration $a_c = v^2/r$
        - Period, time to make one full revolution, $T = \frac{2\pi r}{v}$
    - The force that causes uniform circular motion is always perpendicular to the path the object takes.
    - If the magnitude of the velocity changes, tangential acceleration happens alongside the centripetal acceleration.
        - The centripetal and tangential components, when added, show the direction of $\vec{a}$.
        - ** ONLY HAPPENS WHEN SPEED IS CHANGING
    - For vertical circles, resolve the gravity instead of the tension into x and y components
- Rigid Body Rotation about a Fixed Axis
    - Each particle in the rigid body travels in a circle.
        - Each particle might move at a different speed, but their period T is the same.
    - Angle ("theta"): $\theta(t)$ (rad)
        - $theta = \frac{s}{r}$
        - Radians are really just a ratio between two angles
    - angular velocity ("omega"): $w(t) = \frac{d}{dt}\theta(t)$ (rad/s)
    - angular acceleration ("alpha"): $a(t) = \frac{d}{dt}\omega(t)$ (rad/s^2)
    - $s = r\theta$
    - $v_t = r\omega$
    - $a_t = r\alpha$
    - All kinematic equations for constant $\vec{a}$ function the same for constant α
    - The angular velocity vector is parallel to the axis of rotation
        - Right hand rule, thumb represents angular velocity

Torque

- Torque is what causes rotations
- An unbalanced FORCE causes acceleration, but a net TORQUE causes rotation. Even with no net forces, objects can still rotate ($\vec{a} \neq 0$, object is not in equilibrium)
- All forces produce a torque, τ
- $\tau \propto F, \tau \propto d$
- $\tau = F \times R$ (The force in question times length of the lever)
- A force directed towards the axis of rotation produces no torque.
    - Eg. pushing on a door perpendicular to the hinge
- Only the tangential component of a force produces a torque.

        - As such, $\tau = rF_t = r(F\sin\phi)$, where r is distance to rotation axis and ϕ is angle between r and F
    - Units are $N \cdot m$
    - The shortest distance from the pivot to the force's line of action is the "effective lever arm".
    - $\tau = F\times$ effective lever arm
    - Torque is a vector. It is the cross product of a vector and a scalar.

Cross Product

- $|\vec{C}| = |\vec{A}||\vec{B}|\sin\phi$
- $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$ (not commutative)
- Right hand rule: Position your fingers parallel to A, curl them to B, your thumb indicates the direction of C
- Vector product in cartesian components:
    - $\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$
    - $\vec{i}\times\vec{j} = \vec{k} = -(\vec{j}\times\vec{i})$
    - Remember order ijk
        - i x j = k, j x k = i, k x i = j
- $\vec{\tau} = \vec{r}\times\vec{F}$
- $|\vec{\tau}| = rF\sin\phi$
- Adding torques
    - $\tau_{net} = t_{net,counterclockwise} - \tau_{net,clockwise}$
    - $\tau_{net} =$ sum of τ_i torques (counterclockwise) - sum of τ_j torques (clockwise)

Rigid body equilibriums

- Translational equilibrium: $\vec{F}_{net}\Sigma\vec{F} = 0$
- Rotational equilibrium: $\vec{\tau}_{net}\Sigma\vec{\tau} = 0$
- Therefore, a rigid body is in equilibrium (**static equilibrium**) if both:
    - $\vec{F}_{net} = 0$ and $\vec{\tau}_{net} = 0$
- α is the same for all particles

Moment of inertia

- The angular acceleration of a particle is proportional to the net torque.
- $I = (mr^2)$
- $\tau = I\alpha$
- $\vec{a} = R\cdot\alpha$

- I is the moment of inertia of a particle of mass m relative to the center of motion

Rotations

- $\tau = (mr^2)\alpha$
- Thus $t_{net} = \Sigma_i \vec{\tau} = I$
- Parallel axis theorem
  - The moment of inertia of a body about an axis **parallel** and a distance d away from the center of mass is $\boxed{I = I_{cm} + md^2}$

Center of mass

- Plot all particles onto coordinate plane
- $x_{CM} =$ sum of $m_i x_i$ / sum of m_i
- $x_{CM} =$ sum of $m_i x_i$ / sum of m_i

The center of gravity is $_{CM}$. When considering rotational equilibium of a rigid body, treat the effect of F_g as a force applied at the center of mass.

- Special case: suspension
  - When something is suspended, its center of mass is vertically below the point of suspension
- Only three independent equations can be made from static equilibrium.
  - Two force equations, one torque equation
  - One force equation (components along an axis), two torque equations (torques about two different pivots)
  - Three torque equations (torques about three independent pivots)
  - 2 and 3 are not guaranteed to give independent equations.

Work

- $W_{net} =$ Change in displacement
- Dot product takes two vectors and results in one scalar result
  - $\vec{a} \cdot \vec{b} = \Sigma a_i b_i$ where a_i, b_i
- $W = \vec{s} \cdot \vec{F}$ or $W =$ component of F parallel to motion $\times$ distance
- $\boxed{W = F \cdot d \cdot cos(\theta)}$, F and d are force and displacement, $cos(\theta)$ is angle between F and D
- When an acting force causes displacement, work has been done by the force onto the object.
- **No displacement, no work. The force didn't cause it? No work.**

Variable force work

- Sometimes, the force used to do work is variable. Eg. Squeezig a spring 10cm
- To solve, split displacement into short segments over which F is nearly constant
    - Thus, work is the area under a force vs. distance graph
    - $\boxed{W = \int_{x_1}^{x_f} F\, dx}$

Springs

- Hooke's Law: $\vec{F}_s \propto s$
- The force exerted by the string with stretched in one direction is opposite to the direction of the stretch.
- Work done by a spring:
- Work done by a spring = area under the $F_s \times x$ curve
    - Or $W_s = \frac{1}{2}k(x_i^2 - x_f^2)$ where k is the spring constant
    - Note that work external to the system $\boxed{W_{ext} = \frac{1}{2}k(x_f^2 - x_i^2)}$

Energy

Kinetic Energy

- $\boxed{K = \frac{1}{2}mv^2}$
- $\boxed{K = \frac{1}{2}I\omega^2}$
- As such, $\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = W$
    - Kinetic energy is a scalar, measured in joules (J).
- Work-Kinetic Energy Theorem: The total work done by all external forces acting on a particle is equal to the increase in kinetic energy.
    - Or $W_{ext} = k_f - k_i = \Delta k$
    - Remember that $W = F\Delta x$

Power

- Power is the rate at which work is done
- Average power = Work/time
- Measured in watts W
- Instantaneous power: $P = \vec{F} \cdot \vec{v}$
- Rotational power: $\boxed{P = \tau \cdot \omega}$ and $\boxed{W = \tau\theta}$ for an angular displacement θ

- Remember that $\boxed{\theta = \frac{x}{r}}$ for linear displacement x

Conservative forces are forces where the work done going from A to B is the same for all paths.

- F_g, F_s are usually conservative
- F_f is generally not conservative
- F_t, F_n are uncertain.

Potential energy is energy related to the configuration of a system. It is internal to the system.

Work done by a conservative force is equal to the decrease in potential energy. Eg. A block is lifted. The work done against gravity is stored as **gravitational potential energy** (U_g)

$\boxed{W_{AB} = -\Delta U = U_A - U_B}$

$\boxed{U_g = mgy}$. Units are joules, y is a scalar. Depends only on the vertical height of the object above a surface which you define.

$\boxed{U_s = \frac{1}{2}kx^2}$. The energy (joules) stored inside the compressed spring.

Mechanical Energy

- $E = K + U_g + U_s + \dots.$
- Or, $\boxed{\Delta E_{mech} = \Delta KE + \Delta PE}$
  - Remember that $\Delta KE = W_c + W_{nc}$ and that $\Delta PE = -W_c$ (ONLY FOR CONSERVATIVE FORCES ONLY)
- As long as all forces are conservative, E will be constant no matter what the time is

Momentum

Momentum

- The linear momentum of a particle p is the mass times velocity.
  - $\vec{P} = m\vec{v}$. Units are kg m/s
- Total momentum of a system is vector sum of all momenta in the individual particles
  - $\vec{P}_{total} = \vec{P}_1 + \vec{P}_2 + \dots$
  - Can be broken into x and y components

- $K = \frac{p^2}{2m}$
- $F_{ext} =$ Rate of change of P
  - True even when m is not constant!
- For a constant force:
  - $\boxed{\vec{I} = \Delta\vec{p} = \vec{F}\Delta t}$ (change in momentum = total impulse from external forces)
  - $\vec{I}$ is the impulse.
  - Impulse is area under $\vec{F}$ vs t graph, work is area under $\vec{F}$ vs x graph
- Impulse-momentum theorem:
  - $\boxed{\Delta\vec{p} = \vec{I}}$
  - $\boxed{F_{avg} = \dfrac{\Delta p}{\Delta t}}$
- Conservation of momentum:
  - When internal forces act, momentum is conserved. External forces transfer momentum in or out of the system
  - For an isolated system, $\vec{P}_{f1} + \vec{P}_{f2} = \vec{P}_{i1} + \vec{P}_{i2}$
  - $P_1 + P_2$ is a constant

Collisions

- Types can be elastic, completely inelastic, inelastic
- Completely elastic
  - No sticking at all, bounces away
  - Total kinetic energy is the same before and after
- Inelastic
  - Kinetic energy is converted to other forms
- Completely inelastic
  - Completely sticks together

Motion of a Center of Mass

- Total momentum of a collection of particles is equal to the total mass times the velocity of the center of mass
- $\Sigma_{external forces} = M_{total}\vec{a}_{cm}$
- So $\mathbf{v}_{\mathrm{COM}} = \frac{\sum m_i \mathbf{v}_i}{\sum m_i}$

General Motion of a Rigid Body

- Can be described via translation and rotation of the center of mass rather than from its axis of rotation

- $\Sigma\vec{F} = m\vec{a}_{CM}$
- $\Sigma\tau\vec{_{CM}} = \vec{I}_{CM}\alpha$
- $K = K_{trans} + K_{rot} = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$, use if axis of rotation is not fixed

Rolling Motion

- In pure rolling motion, an object rolls without slipping
- **Combination motion** is when an object rotates about an axis moving on a straight line
- Velocity of any point is the velocity of the center plus tangential velocity of that point relative to center
  - Point on top moves at 2x speed
  - Point on bottom is stationary
- Dynamics:
  - Use $a_{CM} = R\alpha$
  - CM motion: $F = ma_{CM}$
  - Rotation about CM:

Angular Momentum

- $\boxed{L = I\omega}$
  - Units are kg m^2/s
- $\boxed{L = r \times p}$ for the momentum of a particle about an axis
- $\tau_{ext} = I\alpha$ only for a rigid body since I is constant, as such, $F = ma$ only applies when m is constant
- $\tau =$ rate of change of L
- Conservation of angular momentum:
  - When there are no external torques, total angular momentum stays constant.
  - One of the Big Three Conservation Laws, with Cons. of Energy and Cons. of Linear Momentum
- **You can change linear -> angular by changing variables**
  - $m \to I$
  - $v \to \omega$
  - etc.

Springs

Simple Harmonic Motion

- Displacement is a sinusoidal function of time
- $x(t) = A\cos(\omega t + \phi)$
- $\omega t + \phi$ is called the phase, and measured in radians
- The cos function traces one complete cycle when the phase changes by 2π rad
- A is amplitude (maximum and minimum value of x)
- ϕ is phase constant (initial position at $t = 0$)
- ω is related to period T and frequency $\frac{1}{T}$
  - $\boxed{\omega = \dfrac{2\pi}{T}}$, angular frequency
- $\boxed{a = -\omega^2 x}$
- Energy is constant while oscillating

Mass and Springs

- $a = -\frac{k}{m}x$